Sinking Bubbles in an Oscillating Liquid

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March 2014

Abstract

An interesting phenomenon arises in fluid mechanics whereby air bubbles in a liquid can be prevented from rising by the use of oscillations. Several experiments have confirmed that bubbles in a fluid, which is oscillating parallel to the direction of gravity, can sink. We derive the equation of motion for our system and solve for the general motion of a bubble in an oscillating fluid by use of the averaging technique. We conclude that bubbles can sink in an oscillating fluid under certain conditions, and show the conditions on the parameters needed to cause this effect.

1 Introduction

After World War II both the Soviet Union and the United States began a furious campaign to develop liquid fueled rockets. As the space race began, multistage rockets were a primary focus. Although unconfirmed, it can be argued that a problem began to arise where the different stages of the missile would detach during launch tests. Investigations in the U.S. confirmed that the cause was related to fuel level indicators which were designed to trigger the stage separation when propellant levels fell beyond a certain level. The error itself was found to stem from misreadings on this sensor, telling the rocket to release its stages before it was intended. The question was: "How did this sensor misread the fuel levels?"

The root of the problem lies in the realm of the fluid itself, where bubbles in the liquid would disrupt the sensor readings. Fluid dynamics predicts that these bubbles will actually move downwards under certain oscillatory conditions instead of the usual upwards due to buoyancy. Key factors in deciding the motion of these bubbles include the density and pressure of the fluid, the bubble depth, and the frequency of vibration of the liquid.

2 Model Parameters

2.1 Isothermal Bubbles

One generalized assumption made to fit the model was that the bubble size is very small. By assuming that the bubble size is small, the surface area of the bubble begins to dominate the volume of the bubble. Due to this, the bubble is far more likely to be able to diffuse its internal heat into the vibrating fluid. Therefore, one assumes that the heat is fully diffused, and that the bubble temperature is constant.

By using the Ideal Gas Law:

$$P(t)V(t) = P(0)V(0) = Constant \tag{1}$$

Note that the oscillation of the fluid is given by the equation:

$$P_v(t) = A\sin(\omega t) \tag{2}$$

The total pressure on the sphere is therefore given by:

$$P(t) = P_0 + \rho g x + \rho x A\omega^2 \sin(\omega t)$$

Application of equations 1 and 2 imply that the volume of the bubble, as a function of time, is:

$$V(t) = \frac{P(0)V(0)}{P(0) + \rho x(g + A\omega^2 \sin(\omega t))}$$

This is Taylor expanded into the following equation, with the assumption that the Laplace pressure greatly overwhelms the other forces acting on the bubble.

$$V(t) = V(0)[1 - \frac{\rho x g}{P(0)}(1 + \frac{A\omega^2}{g} sin(\omega t))]$$

2.2 Buoyancy Force

Archimedes' Principle states that the buoyant force on an object submerged in water is given by:

$$|F| = \rho V g$$

The upward pressure on a submerged object is equal to the weight of the fluid displaced by the object.

2.3 Drag Force

The drag force on an object moving through a liquid is generally given by:

$$|F_D| = \frac{1}{2}\rho v^2 C_D C_A$$

Here, C_D is the experimentally determined drag coefficient of the particular object in the liquid, C_A is the cross-sectional area, and v is the velocity of the object.

This can be written more generally in the form:

$$F(x') = 4\rho R^2 \Psi(Re) x'^2 sgn(x')$$

R is the radius of the bubble, and $\Psi(Re)$ is the coefficient of resistance, with Re being the Reynolds number as defined by:

$$Re = \frac{2\rho R V}{\mu}$$

with μ being the viscosity of the fluid.

3 Induced Mass

Any object with arbitrary shape will displace a defined quantity of fluid, including bubbles. If it is moving in this fluid, a certain quantity of liquid will be dragged along with the moving bubble, and will increase the effective mass of the bubble.

The kinetic energy of this induced mass, traveling at a velocity U, should be given by the following equation:

$$T = \frac{1}{2}\rho I U^2$$

I is a term that denotes how changes in the velocity of an elemental volume of fluid affects the body associated with the induced mass. This is given by the following equation:

$$I = \int_V ((\frac{u_1}{U})^2 + (\frac{u_2}{U})^2 + (\frac{u_3}{U})^2) dV$$

where u_1, u_2, u_3 represent the componential parts of the fluid velocity.

For a sphere, the potential is given by the dipole potential:

$$\psi = -\frac{UR^3}{2r^2} cos(\phi)$$

Due to the spherical coordinate system for this bubble, the velocities are given by:

$$u_r = \frac{\partial \psi}{\partial r}$$

$$r_\phi = \frac{1}{r}\frac{\partial \psi}{\partial \phi}$$

Substituting the variables in this previous equation into the equation for I gives the following formula:

$$I_{sphere} = \int_V ((\frac{1}{U}\frac{\partial\psi}{\partial r})^2 + (\frac{1}{Ur}\frac{\partial\psi}{\partial\phi})^2)dV$$

This reduces to:

$$I = \frac{2}{3}\pi R^3$$

This is half of the volume of the bubble, so multiplying by the density leads to:

$$m_{att} = \frac{2}{3}\pi\rho R^3$$

This is half of the mass of the liquid displaced by the bubble. Therefore, the attached mass caused by a spherical object moving through a liquid is half the mass of the fluid displaced by the object. Note that in an oscillating fluid, the volume of the bubble will change, causing an oscillatory alteration in the induced mass over time.

4 Governing Equation

After combining the various forces associated with this motion and inserting the model parameters into Newton's Second Law, the governing equation of the bubble system is given by:

$$(m + m_{att})x'' + m'_{att}x' = -F(x') + (m - \rho V(t))(A\omega^2 \sin(\omega t) + g) \tag{3}$$

m_{att} is the attached mass of the bubble, $m'_{att}x'$ is the term associated with the variation of the induced mass, $-F(x')$ represents the drag force, and the last term is associated with the buoyancy force and the pressure fluctuations in the vibrating liquid.

5 Separation of Variables

In a common method used in harmonic analysis, the method of separation of variables is used.

$$x(t) = X(t) + \psi(t)$$

Here, $X(t)$ is the equation of 'slow motion,' or also known as the equation of general motion (i.e. whether the bubble moves upward, downward, or is stationary). Ψ is the equation of 'fast motion,' so it describes the fast oscillating terms in the model.

The oscillatory terms are considered the fast terms in the equation of motion. These all have a time average motion of 0, for a uniformly oscillating liquid, and therefore are not accounted for in the general motion of the system.

Equation 3 is then separated into the 'fast' and 'slow' equations of motion. The fast solution is shown in equation 4, and the slow solution in equation 5.

$$(m + m_{att})\Psi'' = -4\rho R_0^2(Re)(\Psi'^2 sgn\Psi' - (\Psi'^2 sgn\Psi')) + (m - \rho V(0))A\omega^2 sin(\omega t)) \tag{4}$$

The 'slow' solution is show here as:

$$(m + m_{att})X'' + \langle F(X' + \Psi')\rangle = \gamma\omega^2\frac{\rho V(0)g}{2}\frac{X}{H_0}(1 - \frac{2}{3}(1 - \frac{m}{\rho V(0)})sin^2(\phi)) - (\rho V(0) - m)g \tag{5}$$

where $\gamma = \frac{\rho H_0 g}{P(0)}$. H_0 is defined as the height of the fluid column.

6 Averaging Procedure

The separation of variables indicates that the solution is going to be a sum of two time scaled functions represented by the following equation:

$$\langle x(t)\rangle = \langle X(t)\rangle + \langle\Psi(\tau)\rangle$$

where $X(t)$ is the slow time scale function (represents the trajectory of the bubble while sinking) and, $\Psi(\tau)$ is the fast time scale function (represents the oscillation of the bubble). Since the fast time scale function $\Psi(\tau)$ is periodic, it has an average value of 0 over one period. On the other hand, the slow time scale function has an average value which is non-zero. Therefore, the whole trajectory of the bubble has an average value that is equal to the average value of the slow time scale function $X(t)$. So, the average equation ends up as the following:

$$\langle x(t) \rangle = \langle X(t) \rangle$$

Thus, we can see that the whole trajectory of the bubble is described by the slow time scale function.

7 Conclusion

We then assume that the acceleration term is relatively small and remove the second order terms (both are due to the small size of the bubbles). We consider the mass of the bubble to be relatively small in comparison to the induced mass of the system, i.e. $m << m_{att}$. After applying these approximations and assumptions to the 'slow' solution (equation 5), we arrive at the general equation of motion for the system:

$$X'(t) = \nu[\frac{X}{X_0} - 1]$$

Where X_0 is given by equation 6. Here, R_0 is the initial radius of the bubble, γ was previous mentioned, H_0 is the fluid height, and θ is equal to $\frac{16^2 Re^2}{\pi^4 \chi^2}$. The value of ν is equal to $\frac{\pi^2}{12Re} \frac{R_0 g}{B\omega}$.

$$X_0 = \frac{2H_0}{\gamma\omega^2} \frac{2(1+\sqrt{(}1+\theta\frac{A^2}{R_0^2})+\theta\frac{A^2}{R_0^2}}{2(1+\sqrt{(}1+\theta\frac{A^2}{R_0^2}))+\frac{\theta}{3}\frac{A^2}{R_0^2}} \quad (6)$$

For $X > X_0$, we see that the bubble rises, whereas for $X < X_0$, the bubble will sink. For the condition $X = X_0$, the bubble will remain neither sink nor rise.

We conclude that bubbles in an oscillating liquid can sink under certain conditions. Figure 1 contains a plot of the fast and slow solutions. The solid line represents the 'fast solution,' while the dotted line represents the 'slow solution.'

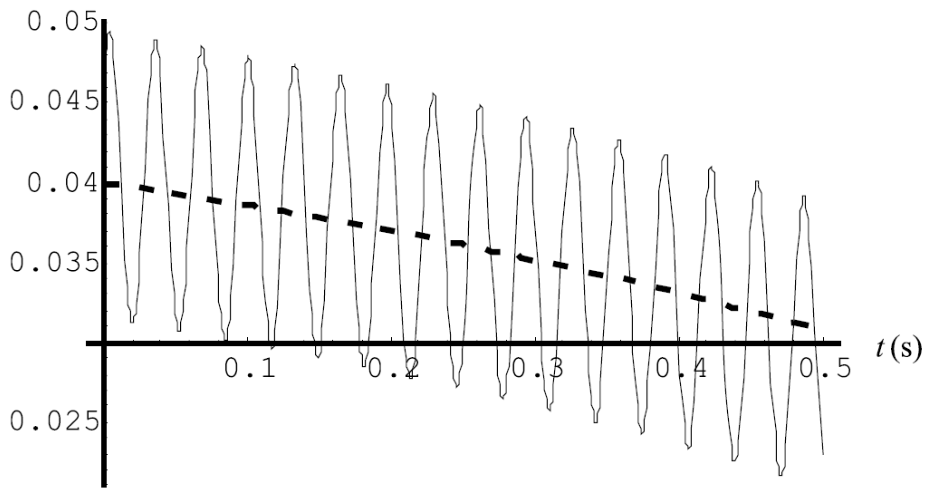

Figure 1: Displacement (m) vs. Time (s) - Source (3)

8 Future Work

We intend to verify the final slow equation using computational modeling. The ODE solution of the system will be approximated computational using either C++ or Matlab, and we will verify if the assumptions made in order to approximate the final solution function were sufficient. We will determine the conditions of stability of the solutions, and create a bifurcation diagram to demonstrate this.

We will then explore one of two possible routes. The first would be to determine if the condition on the bubbles sinking can be changed using a device such as an ultrasound generator. We would determine if it is possible to stop these bubbles from sinking with an applicable way to change the oscillations of the fluid.

The second possible route of study would be to model the addition of thermal excitations on the bubbles themselves. The pressure and volume of the bubbles would not be able to be modeled as an isothermic system, and the pressure and volume of the bubble with respect to time would change. We would include this excitation term in our ODE, and determine its effect on the general motion of the bubble.

9 Sources

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